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APPLICATIONS OF DERIVATIVES

IMPORTANT FORMULAE

1. Rate of Change : If two quantities x and y vary with respect to another quantity θ *i.e.*, if $x = f(\theta)$ and $y = g(\theta)$, then by chain rule :

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}, \text{ if } \frac{dx}{d\theta} \neq 0$$

2. Equation of Tangent : Equation of tangent line on the point P (x_1, y_1) of curve $y = f(x)$

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

3. Equation of Normal : Equation of normal at point $P(x_1, y_1)$ of curve $y = f(x)$

4. A function f:

(a) is increasing in interval $[a, b]$ if in (a, b)

$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in (a, b)$

(b) is decreasing in interval $[a, b]$ if in (a, b)

$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in (a, b)$

- Let $y = f(x)$ and Δx be small increment in x and Δy be increment in y correspond of increment of x, *i.e.*, $\Delta y = f(x + \Delta x) - f(x)$, then

$$dy = f'(x)\,dx \text{ or } dy = \left(\frac{dy}{dx}\right)\Delta x$$

when $dx = \Delta x$ is relatively small when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

- A point c in domain of a function f, at which $f'(c) = 0$ or f is not differentiable is called a critical point of f.
- **Working Rule :**

First Derivative Test :

1. Find $\frac{dy}{dx}$ and evaluate real values of x after equating it to 0.
2. Write the sign scheme of $\frac{dy}{dx}$.
3. The interval in which $\frac{dy}{dx}$ is positive, y will be increasing function and the interval in which $\frac{dy}{dx}$ is negative, y will be decreasing function.
4. If $f(x)$ is continuous, then $f(x)$, *i.e.*, value of y is maximum at $x = a$. If the a function $f(x)$ is increasing to the left and decreasing to the right of $x = a$.

Second Derivative Test :

1. To find the maximum and minimum value of a function $f(x)$ write $y = f(x)$.
2. Find $\frac{dy}{dx}$ and find the real value of x equating it to 0.
3. If $\frac{dy}{dx} = 0$ does not give any real value of x, then the value of y will neither be maximum nor minimum.
4. If $\frac{dy}{dx} = 0$ gives real values α, β, λ, etc. of x, then find second derivative $\frac{d^2y}{dx^2}$ of y at these points.
5. Value of y at that point :

(i) will be maximum if $\frac{d^2y}{dx^2} < 0$.

(ii) will be minimum if $\frac{d^2y}{dx^2} > 0$.

- If $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$, then these will be neither maximum nor minimum at that point such a point is called the point of inflexion.

➠ Multiple Choice Questions

1. The normal to a given curve is parallel to x-axis if : (*BSEB*, 2010)

(a) $\frac{dy}{dx} = 0$ (b) $\frac{dy}{dx} = 1$

(c) $\frac{dx}{dy} = 0$ (d) $\frac{dx}{dy} = 1$

2. The maximum value of $\left(\frac{1}{x}\right)^x$ is : (*BSEB*, 2010)

(a) 1 (b) e

(c) $e^{\frac{1}{e}}$ (d) none of these

3. The radius of a circle is increasing at the rate of 0.4 cm/sec. The rate of increase of its circumference is (*BSEB*, 2010)

(a) 0.4 cm/sec (b) 0.8π cm/sec

(c) 0.8 cm/sec (d) none of these

4. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is :

(a) $(2\sqrt{2}, -1)$ (b) $(2\sqrt{2}, 0)$

(c) (0, 0) (d) (2, 2)

5. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point :

(a) (1, 2) (b) (2, 1)

(c) (1, – 2) (d) (– 1, 2)

6. The interval in which $y = x^2 e^{-x}$ is increasing with respect to x is :

(a) $(-\infty, \infty)$ (b) (– 2, 0)

(c) $(2, \infty)$ (d) (0, 2)

7. The least value of a such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing in (1, 2) is :
(a) -2 (b) -3 (c) 0 (d) 1

8. The curves $x = y^2$ and $xy = k$ cut at right angle if $8k^2 =$
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

9. The tangents to the curve $y = 7x^2 + 11$ at the points, where $x = 2$ and $x = -2$ are :
(a) parallel (b) perpendicular
(c) coincident (d) none of these

10. The local maximum value of the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$ is :
(a) 8 (b) 10 (c) 12 (d) 16

Ans. 1. (c), 2. (c), 3. (b), 4. (a), 5. (a), 6. (d), 7. (a), 8. (a), 9. (a), 10. (c).

Very Short Answer Type Questions

Q. 1. Find the slope of the tangent to the curve $y = x^3 - 2x + 8$ at the point (1, 7). (*JAC*, 2014)

Solution :

$$y = x^3 - 2x + 8$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 2$$

$$\text{at } (1, 7), \quad \frac{dy}{dx} = 3(1)^2 - 2 = 1$$

Hence, the required slope is 1.

Q. 2. The total expenditure (in ₹) required for providing the cheap edition of a book for poor and deserving students is given by $R(x) = 3x^2 + 36x$, where x is the number of sets of books. If the marginal expenditure is defined as $\frac{dR}{dx}$, write he marginal expenditure required for 1200 such sets. What value is reflected in this question ? [*CBSE*, 2013 (*Comptt.*)]

Solution :

$$R(x) = 3x^2 + 36x$$

$$\Rightarrow \quad \frac{dR}{dx} = 6x + 36$$

For $x = 1200$,

$$\frac{dR}{dx} = 6\,(1200) + 36 = 7{,}236$$

Hence, the marginal expenditure required for 1200 such sets is ₹ 7,236.

Value reflected

Marginal expenditure increases with increase in the number of sets of books.

Q. 3. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupee) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$, and write which value does the question indicate. (*AI CBSE*, 2013)

Solution :

Total revenue $\{R(x)\} = 3x^2 + 36x + 5$

$$\therefore \quad \text{Marginal revenue} = \frac{d}{dx}\{R(x)\} = 6x + 36$$

at $x = 5$,

$$\text{Marginal revenue} = 6(5) + 36 = 66$$

Value Indicated

More amount of money is spent for the welfare of the employees with the increase in marginal revenue.

Q. 4. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above question. (*CBSE*, 2013)

Solution :

We have

$$P(x) = 0.005x^3 + 0.02x^2 + 30x$$

$$\Rightarrow \quad \frac{d}{dx}\{P(x)\} = 0.015x^2 + 0.04x + 30$$

at $x = 3$,

$$\frac{d}{dx}\{P(x)\} = 0.015\,(3)^2 + 0.04(3) + 30$$
$$= 0.135 + 0.12 + 30$$
$$= 30.255$$

Value indicated

The pollution content in the city increases with the increase in the number of diesel vehicles.

Q. 5. A stone is dropped into a quiet lake and the waves move in circles. If the radius of a circular wave increases at the rate of 5 cm/sec, find the rate of increase in its area at the instant when its radius is 8 cm. (*JAC*, 2014)

Solution :

Let the radius of a wave be r cm and the area be A cm^2 at time t, then

$$A = \pi r^2$$

$$\frac{dr}{dx} = 5 \quad \text{(Given)}$$

$$\therefore \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$
$$= 2\pi r\,(5)$$
$$= 10\pi r$$

when $r = 8$ cm,

$$\frac{dA}{dt} = 10\pi\,(8) = 80\pi \text{ cm}^2/\text{sec}$$

Q. 6. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area increases, when the side is 10 cm. [*AI CBSE*, 2014 (*Comptt.*)]

Solution :

Let a cm be the side of an equilateral triangle, then

$$\frac{da}{dt} = 2 \text{ cm/sec} \qquad \text{(Given) ...(1)}$$

Area of equilateral triangle (A)

$$= \frac{\sqrt{3}}{4} a^2$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4} 2a \frac{da}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4} a(2) \qquad \text{[Using (1)]}$$

$$\Rightarrow \quad \frac{dA}{dt} = \sqrt{3}\, a$$

when $a = 10$ cm

$$\frac{dA}{dt} = \sqrt{3}\ (10)\ \text{cm}^2/\text{sec}^2$$

$$\Rightarrow \quad \frac{dA}{dt} = 10\sqrt{3}\ \text{cm}^2/\text{sec}^2$$

➠ Short Answer Type Questions

Q. 1. Using differentials, find the approximate value of $(3.968)^{3/2}$. [*CBSE*, 2014 (*Comptt.*)]

Solution :

Let $y = x^{3/2}$

then $y + \delta y = (x + \delta x)^{3/2}$

Subtracting, we get

$$\delta y = (x + \delta y)^{3/2} - x^{3/2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)\delta x = (x + \delta x)^{3/2} - x^{3/2}$$

$\left[\because dy = \left(\frac{dy}{dx}\right)\delta x\right.$ is approximately equal to δy

$$\Rightarrow \quad \frac{3}{2} x^{\frac{1}{2}} \sqrt{x} = (x + \delta x)^{3/2} - x^{3/2} \qquad \ldots(1)$$

Let $x = 4$ and $\sqrt{x} = -0.032$

then (1) gives

$$\frac{3}{2}(4)^{\frac{1}{2}}(-0.032) = (4 - 0.032)^{3/2} - (4)^{3/2}$$

$$\Rightarrow \quad -0.096 = (3.968)^{3/2} - 8$$

$$\Rightarrow \quad (3.968)^{3/2} = 8 - 0.096$$

$$\Rightarrow \quad (3.968)^{3/2} = 7.904$$

Q. 2. Prove that the function f given by $f(x)$ = log cos x is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$. (*BSER*, 2014)

Solution :

$$f(x) = \log \cos x$$

$$\Rightarrow \quad f'(x) = -\frac{\sin x}{\cos x} = -\tan x$$

on $\left(0, \frac{\pi}{2}\right)$, $\tan x > 0$

$\Rightarrow \quad -\tan x < 0$

$\Rightarrow \quad f'(x) < 0$

$\therefore f(x)$ is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$.

in $\left(\frac{\pi}{2}, \pi\right)$, $\tan x < 0$

$\Rightarrow \quad -\tan x > 0$

$\Rightarrow \quad f'(x) > 0$

$\therefore f(x)$ is strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$.

Q. 3. Prove that the function $f(x)$ = cos x is :

(a) strictly decreasing in $(0, \pi)$

(b) strictly increasing in $(\pi, 2\pi)$. (*USEB*, 2013)

Solution :

(a) $f(x) = \cos x$

$\Rightarrow \quad f'(x) = -\sin x$

In $(0, \pi)$, $\sin x > 0$

$\Rightarrow \quad -\sin x < 0$

$\Rightarrow \quad f'(x) < 0$

$\therefore f(x)$ in strictly decreasing in $(0, \pi)$.

(b) $f(x) = \cos x$,

$\Rightarrow \quad f'(x) = -\sin x$

In $(\pi, 2\pi)$, $\sin x < 0$

$\Rightarrow \quad -\sin x > 0$

$\therefore f(x)$ in strictly decreasing in $(\pi, 2\pi)$.

Q. 4. Find the intervals in which the function given by $f(x) = x^2 - 4x + 6$ is :

(a) strictly increasing

(b) strictly decreasing. (*USEB*, 2014)

Solution :

$$f(x) = x^2 - 4x + 6$$

$$\Rightarrow \quad f'(x) = 2x - 4$$

(a) If $f(x)$ is strictly increasing, then

$f'(x) > 0$

$\Rightarrow \quad 2x - 4 > 0$

$\Rightarrow \quad 2(x - 2) > 0$

$\Rightarrow \quad x - 2 > 0$

$\Rightarrow \quad x > 2$

$\therefore f(x)$ is strictly increasing in $(2, \infty)$

(b) If $f(x)$ in strictly decreasing, then

$f'(x) < 0$

$\Rightarrow \quad x < 2$

$\therefore f(x)$ is strictly decreasing in $(-\infty, 2)$

Q. 5. Find the intervals in which the function given by $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is :

(a) strictly increasing

(b) strictly decreasing. [*AI CBSE*, 2014 (*Comptt.*)]

Solution :

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

$$\Rightarrow f'(x) = \frac{3}{10}(4x^3) - \frac{4}{5}(3x^2) - 3(2x) + \frac{36}{5}$$

$$\Rightarrow f'(x) = \frac{6}{5}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$$

$$\Rightarrow f'(x) = \frac{6}{5}(x^3 - 2x^2 - 5x + 6)$$

$$\Rightarrow f'(x) = \frac{6}{5}(x-1)(x-3)(x+2)$$

$$\Rightarrow f'(x) = \frac{6}{5}(x+2)(x-1)(x-3)$$

$$f'(x) = 0$$

$$\Rightarrow \quad \frac{6}{5}(x+2)(x-1)(x-3) = 0$$

$\Rightarrow \quad (x + 2)(x - 1)(x - 3) = 0$

$\Rightarrow \quad x = -2, 1, 3$

$\therefore$ The critical values of x in the ascending order are $x = -2, 1, 3$

$x < -2$

$\Rightarrow \quad f'(x) = \frac{6}{5}(-)(-)(-) = -ve$

$\Rightarrow$ $f(x)$ is strictly decreasing in $(-\infty, -2)$.

$-2 < x < 1$

$\Rightarrow \quad f'(x) = \frac{6}{5}(+)(-)(-) = +ve$

$\Rightarrow f(x)$ is strictly increasing in $(-2, 1)$.

$1 < x < 3$

$\Rightarrow \quad f'(x) = (+)(+)(-) = -ve$

$\Rightarrow f(x)$ is strictly decreasing in $(1, 3)$.

$3 < x < \infty$

$\Rightarrow \quad f'(x) = \frac{6}{5}(+)(+)(+) = +ve$

$\Rightarrow$ $f(x)$ is strictly increasing in $(3, \infty)$.

Hence $f(x)$ is :

(a) Strictly increasing in $(-2, 1) \cup (3, \infty)$

(b) Strictly decreasing in $(-\infty, -2) \cup (1, 3)$

Q. 6. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is :

(a) Strictly increasing

(b) Strictly decreasing. (*CBSE*, 2014)

Solution :

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$\Rightarrow \quad f'(x) = 12x^3 - 12x^2 - 24x$

$f'(x) = 0$

$\Rightarrow \quad 12x^3 - 12x^2 - 24x = 0$

$\Rightarrow \quad 12x(x^2 - x - 2) = 0$

$\Rightarrow \quad 12x(x - 2)(x + 1) = 0$

$\Rightarrow \quad x(x - 2)(x + 1) = 0$

$\Rightarrow \quad (x + 1)x(x - 2) = 0$

$\Rightarrow \quad x = -1, 0, 2$

$\therefore$ The critical values of x in ascending order are $x = -1, 0, 2$

$x < -1$

$\Rightarrow \quad f'(x) = 12(-)(-)(-) = -ve$

$\Rightarrow \quad f(x)$ is strictly decreasing in $(-\infty, -1)$,

$-1 < x < 0$

$\Rightarrow \quad f'(x) = 12(-)(-)(+) = +ve$

$\Rightarrow \quad f(x)$ is strictly increasing in $(-1, -0)$.

$0 < x < 2$

$\Rightarrow \quad f'(x) = 12(+)(-)(+) = -ve$

$\Rightarrow \quad f(x)$ is strictly decreasing in $(0, 2)$.

$2 < x < \infty$

$\Rightarrow \quad f'(x) = 12(+)(+)(+) = +ve$

$\Rightarrow \quad f(x)$ is strictly increasing in $(2, -\infty)$

Hence

(a) $f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$

(b) $f(x)$ is strictly decreasing in

$$(-\infty, -1) \cup (0, 2)$$

Q. 7. Find the intervals in which $f(x) = (x - 1)(x - 2)^2$; $x \in$ R is :

(a) increasing

(b) decreasing. (*JAC*, 2014)

Solution :

$f(x) = (x - 1)(x - 2)^2$

$\Rightarrow \quad f'(x) = (x - 2)^2 + 2(x - 1)(x - 2)$

$\Rightarrow \quad f'(x) = (x - 2)(x - 2 + 2x - 2)$

$\Rightarrow \quad f'(x) = (x - 2)(3x - 4)$

(a) If $f(x)$ is increasing, then

$$f'(x) > 0$$

$\Rightarrow \quad (x - 2)(3x - 4) > 0$

$\Rightarrow \quad (x - 2)\left(x - \frac{4}{3}\right) > 0$

$\Rightarrow \quad x < \frac{4}{3}$ or $x > 2$

$\therefore f(x)$ is increasing on $\left(-\infty, \frac{4}{3}\right)$ and $(2, \infty)$.

(b) If $f(x)$ is decreasing, then

$$f'(x) < 0$$

$\Rightarrow \quad (x - 2)\left(x - \frac{4}{3}\right) < 0$

$\Rightarrow \quad \frac{4}{3} < x < 2$

$\therefore f(x)$ is decreasing on $\left(\frac{4}{3}, 2\right)$.

Q. 8. Find the values of x for which $y = [x(x - 2)]^2$ is an increasing function. (*AI CBSE*, 2014)

Solution :

$y = [x(x - 2)]^2$

$\Rightarrow \quad y = (x^2 - 2x)^2$

$\Rightarrow \quad y = x^4 - 4x^3 + 4x^2$

$\Rightarrow \quad \frac{dy}{dx} = 4x^3 - 12x^2 + 8x$

$\Rightarrow \quad \frac{dy}{dx} = 4x(x^2 - 3x + 2)$

$\Rightarrow \quad \frac{dy}{dx} = 4x(x - 1)(x - 2)$

$\frac{dy}{dx} = 0$

$\Rightarrow \quad 4x(x - 1)(x - 2) = 0$

$\Rightarrow \quad x(x - 1).(x - 2) = 0$

$\Rightarrow \quad x = 0, 1, 2$

$\therefore$ The critical values of x in ascending order are 0, 1, 2.

$x < 0$

$\Rightarrow \quad \frac{dy}{dx} = 4(-)(-)(-) = -ve$

$\Rightarrow y$ is strictly decreasing in $(-\infty, 0)$.

$0 < x < 1$

$\Rightarrow \quad \frac{dy}{dx} = 4(+)(-)(-) = +ve$

$\Rightarrow y$ is strictly increasing in $(0, 1)$.

$1 < x < 2$

$\Rightarrow \quad \frac{dy}{dx} = 4(+)(+)(-) = -ve$

$\Rightarrow y$ is strictly decreasing in $(1, 2)$.

$x > 2$

$\Rightarrow \quad \frac{dy}{dx} = 4(+)(+)(+) = +ve$

$\Rightarrow y$ is strictly increasing in $(2, \infty)$.

Hence, y is increasing on $(0, 1)$ and $(2, \infty)$.

Q. 9. Find the maximum and minimum values of the function $f(x) = 2x^3 - 15x^2 + 36x + 11$.

(*JAC*, 2013)

Solution :

$$f(x) = 2x^3 - 15x^2 + 36x + 11$$
$$\Rightarrow \quad f'(x) = 6x^2 - 30x + 36$$

For maxima or minima,

$$f'(x) = 0$$
$$\Rightarrow \quad 6x^2 - 30x + 36 = 0$$
$$\Rightarrow \quad 6(x^2 - 5x + 6) = 0$$
$$\Rightarrow \quad x^2 - 5x + 6 = 0$$
$$\Rightarrow \quad (x-2)(x-3) = 0$$
$$\Rightarrow \quad x = 2, 3$$

$\therefore$ Critical points are $x = 2$ and $x = 3$.

$$f''(x) = 12x - 30$$

At $x = 2$, $\quad f''(x) = 12(2) - 30 = -6 < 0$

$\therefore$ $x = 2$ is a point of maxima.

Maximum value $= f(2)$
$= 2(2)^3 - 15(2)^2 + 36(2) + 11$
$= 16 - 60 + 72 + 11$
$= 39$

At $x = 3$, $\quad f''(x) = 12(3) - 30$
$= 6 > 0$

$\therefore$ $x = 3$ is a point of minima.

Minimum value $= f(3)$
$= 2(3)^3 - 15(3)^2 + 36(3) + 11$
$= 54 - 135 + 108 + 11 = 38$

Q. 10. Find the equation of the tangent and normal to the curve $x^{2/3} + y^{2/3} = 2$ at the point (1, 1). *(USEB, 2014)*

Solution :

The equation of the curve is

$$x^{2/3} + y^{2/3} = 2$$

Differentiating w.r.t. x, we get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

at the point (1, 1),

$$\frac{dy}{dx} = -\frac{1^{\frac{1}{3}}}{1^{\frac{1}{3}}} = -1$$

$\therefore$ Slope of the tangent at (1, 1) $= -1$

$\therefore$ Equation of the tangent at the point (1, 1) is

$$y - 1 = -1(x-1)$$
$$\Rightarrow \quad y - 1 = -x + 1$$
$$\Rightarrow \quad y + x = 2$$

Slope of the normal at (1, 1) $= -\dfrac{1}{(-1)} = 1$

$(\because m_1 m_2 = -1)$

$\therefore$ Equation of the normal at the point (1, 1) is

$$y - 1 = 1(x-1)$$
$$\Rightarrow \quad y = x$$

Q. 11. Find the equation of the tangent and normal to the curve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $(\sqrt{2}\,a, b)$ *(AI CBSE, 2014)*

Solution :

The equation of the curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

at $(\sqrt{2}\,a, b)$, $\quad \dfrac{dy}{dx} = \dfrac{b^2\sqrt{2}a}{a^2 b} = \dfrac{b\sqrt{2}}{a}$

$\therefore$ Equation of the tangent at the point $(\sqrt{2}\,a, b)$ is

$$y - b = \frac{b\sqrt{2}}{a}(x - \sqrt{2}\,a)$$
$$\Rightarrow \quad ay - ab = b\sqrt{2}\,x - 2ab$$
$$\Rightarrow \quad b\sqrt{2}\,x - ay - ab = 0$$

Slope of normal at $(\sqrt{2}\,a, b) = -\dfrac{1}{\left(\dfrac{b\sqrt{2}}{a}\right)} = -\dfrac{a}{b\sqrt{2}}$

$\therefore$ Equation of normal at the point $(\sqrt{2}\,a, b)$ is

$$y - b = -\frac{a}{b\sqrt{2}}(x - \sqrt{2}\,a)$$
$$\Rightarrow \quad b\sqrt{2}\,y - b^2\sqrt{2} = -ax + \sqrt{2}\,a^2$$
$$\Rightarrow \quad ax + b\sqrt{2}\,y - \sqrt{2}(a^2 + b^2) = 0$$

Q. 12. Find the equations of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \dfrac{\pi}{4}$. *(CBSE, 2014)*

Solution :

$$x = a\sin^3\theta$$
$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

and

$$y = a\cos^3\theta$$
$$\Rightarrow \quad \frac{dy}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\cot\theta$$

at $\theta = \dfrac{\pi}{4}$, $\quad \dfrac{dy}{dx} = -\cot\dfrac{\pi}{4} = -1$

$\therefore$ Slope of tangent at $\theta = \dfrac{\pi}{4} = -1$

$$x = a\sin^2\frac{\pi}{4} = a\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{a}{2\sqrt{2}}$$

$\therefore$ Point of contact is $\left(\dfrac{a}{2\sqrt{2}}, \dfrac{a}{2\sqrt{2}}\right)$.

$\therefore$ Equation of tangent at $\theta = \dfrac{\pi}{4}$ is

$$y - \frac{a}{2\sqrt{2}} = -1\left(x - \frac{a}{2\sqrt{2}}\right)$$

$$\Rightarrow \quad y + x = \frac{a}{\sqrt{2}}$$

Slope of normal at $\theta = \frac{\pi}{4} = -\frac{1}{(-1)} = 1$

$\therefore$ Equation of normal at $\theta = \frac{\pi}{4}$ is

$$y - \frac{a}{2\sqrt{2}} = x - \frac{a}{2\sqrt{2}}$$

$$\Rightarrow \quad y - x = 0$$

Q. 13. Find the equations of the tangent and normal to the curve $x = \sin 3t$, $y = \cos 2t$ at the point $t = \frac{\pi}{4}$. (*JAC*, 2014)

Solution :

We have

$$x = \sin 3t$$
$$y = \cos 2t$$

at $t = \frac{\pi}{4}$, $\quad x = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

$$y = \cos\frac{\pi}{2} = 0$$

$\therefore$ The point of contact is $\left(\frac{1}{\sqrt{2}}, 0\right)$.

$$\frac{dx}{dt} = 3\cos 3t$$

and $\quad \frac{dy}{dt} = -2\sin 2t$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{3\cos 3t}$

at $t = \frac{\pi}{4}$, $\quad \frac{dy}{dx} = \frac{-2\sin\frac{\pi}{2}}{3\cos\frac{3\pi}{4}}$

$$= \frac{-2(1)}{3\left(-\frac{1}{\sqrt{2}}\right)} = \frac{2\sqrt{2}}{3}$$

$\therefore$ Slope of tangent at $\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{2\sqrt{2}}{3}$

$\therefore$ Equation of tangent at $\left(\frac{1}{\sqrt{2}}, 0\right)$ is

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y = \frac{2\sqrt{2}}{3}x - \frac{2}{3}$$

$$\Rightarrow \quad 3y = 2\sqrt{2}\,x - 2$$

Slope of normal at $\left(\frac{1}{\sqrt{2}}, 0\right)$ is

$$= -\frac{1}{\left(\frac{2\sqrt{2}}{3}\right)} = -\frac{3}{2\sqrt{2}}$$

$\therefore$ Equation of normal at $\left(\frac{1}{\sqrt{2}}, 0\right)$ is

$$y - 0 = -\frac{3}{2\sqrt{2}}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y = -\frac{3}{2\sqrt{2}}x + \frac{3}{4}$$

$$\Rightarrow \quad 4y = -3\sqrt{2}\,x + 3$$

Q. 14. Find the points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are parallel to x-axis. (*BSER*, 2014)

Solution :

$$\frac{x^2}{4} + \frac{y^2}{25} = 1 \qquad \ldots(1)$$

Differentiating w.r.t. x, we get

$$\frac{2x}{4} + \frac{2y}{25}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{25}{4}x$$

If the tangent line is parallel to x-axis, then

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad -\frac{25}{4}x = 0$$

$$\Rightarrow \quad x = 0$$

Putting $x = 0$ in (1), we get

$$\frac{y^2}{25} = 1$$

$$\Rightarrow \quad y = \pm 5$$

Hence at the points $(0, \pm 5)$, the tangent lines are parallel to x-axis.

Q. 15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(i) parallel to the line $2x - y + 9 = 0$

(ii) perpendicular to the line $5y - 15x = 13$.

[*CBSE*, 2014 (*Comptt.*)]

Solution :

(i) The equation of the curve is

$$y = x^2 - 2x + 7 \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x - 2 \qquad \ldots(2)$$

Equation of line is

$$2x - y + 9 = 0 \qquad \ldots(3)$$

$$\Rightarrow \quad y = 2x - 9$$

$\therefore$ Slope of (3) is 2.

$\therefore$ Slope of the tangent line = 2

$$\Rightarrow \quad \frac{dy}{dx} = 2$$

$$\Rightarrow \quad 2x - 2 = 2$$

$$\Rightarrow \quad 2x = 4$$

$$\Rightarrow \quad x = 2$$

From (1), $\quad y = 2^2 - 2(2) + 7 = 7$

Hence the points of contact is (2, 7).

$\therefore$ Equation of the tangent line is

$$y - 7 = 2(x - 2)$$

$$\Rightarrow \quad y = 2x + 3$$

(ii) The given line is

$$5y - 15x = 13 \quad \ldots(4)$$

$$\Rightarrow \quad 5y = 15x + 13$$

$$\Rightarrow \quad y = 3x + \frac{13}{5}$$

$\therefore$ Slope of line $5y - 15x = 13$ is 3.

$\because$ The tangent line is perpendicular to the line (4),

$$\therefore \quad \frac{dy}{dx}(3) = -1$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{3}$$

$$\Rightarrow \quad 2x - 2 = -\frac{1}{3}$$

$$\Rightarrow \quad 6x - 6 = -1$$

$$\Rightarrow \quad 6x = 5$$

$$\Rightarrow \quad x = \frac{5}{6}$$

From (1),

$$y = \left(\frac{5}{6}\right)^2 - 2\left(\frac{5}{6}\right) + 7$$

$$= \frac{25}{36} - \frac{5}{3} + 7$$

$$= \frac{25 - 60 + 252}{36} = \frac{217}{36}$$

Hence the point of contact is

$$\left(\frac{5}{6}, \frac{217}{36}\right)$$

$\therefore$ Equation of the tangent line is

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$

$$\Rightarrow \quad 36y - 217 = -12x + 10$$

$$\Rightarrow \quad 12x + 36y = 227$$

Q. 16. Find the equations of the normals to the curve $2x^2 - y^2 = 14$ which are parallel to the line $x + 3y = 6$. *(BSER, 2013)*

Solution :

The given line is

$$x + 3y = 6$$

$$\Rightarrow \quad 3y = -x + 6$$

$$\Rightarrow \quad y = -\frac{x}{3} + 2$$

$$\therefore \quad \text{Slope of line} = -\frac{1}{3}.$$

The given curve is

$$2x^2 - y^2 = 14$$

Differentiating w.r.t. x, we get

$$4x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2x = y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{y}$$

Let the normal at (h, k) be parallel to the given line, then

$$\text{slope of tangent at } (h, k) = \left.\frac{dy}{dx}\right|_{(h,k)} = \frac{2h}{k}$$

$$\therefore \text{ Slope of normal at } (h, k) = -\frac{1}{\left(\frac{2h}{k}\right)} = \frac{-k}{2h}$$

According to the question,

$$-\frac{k}{2h} = -\frac{1}{3}$$

$$\Rightarrow \quad 2h = 3k$$

Again, (h, k) lies on the curve $2x^2 - y^2 = 14$

$$\therefore \quad 2h^2 - k^2 = 14$$

$$\Rightarrow \quad 2h^2 - \frac{4h^2}{9} = 14 \quad \left(\because 2h = 3k \Rightarrow k = \frac{2h}{3}\right)$$

$$\Rightarrow \quad \frac{14h^2}{9} = 14$$

$$\Rightarrow \quad h^2 = 9$$

$$\Rightarrow \quad h = \pm 3$$

$$\Rightarrow \quad k = \pm 2$$

$\therefore$ The points are $(3, 2)$ and $(-3, -2)$.

Normal at (3, 2)

$$\text{Slope of normal} = -\frac{k}{2h}$$

$$= -\frac{2}{2\times3} = -\frac{1}{3}$$

$\therefore$ Equation of normal is

$$y - 2 = -\frac{1}{3}(x - 3)$$

$$\Rightarrow \quad 3y - 6 = -x + 3$$

$$\Rightarrow \quad x + 3y = 9$$

Normal at (– 3, – 2)

$$\text{Slope of normal} = -\frac{k}{2h} = \frac{-(-2)}{2(-3)} = -\frac{1}{3}$$

$\therefore$ Equation of normal is

$$y + 2 = -\frac{1}{3}(x + 3)$$

$$\Rightarrow \quad 3y + 6 = -x - 3$$

$$\Rightarrow \quad x + 3y + 9 = 0$$

Q. 17. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $\left(\frac{4}{3}, 0\right)$. *(AI CBSE, 2013)*

Solution :

The equation of the curve is

$$3x^2 - y^2 = 8 \quad \ldots(1)$$

Differentiating w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x}{y}$$

Let the point of contact be (h, k), then

$$\left.\frac{dy}{dx}\right|_{at\ (h,k)} = \frac{3h}{k}$$

$\therefore$ Equation of the tangent at (h, k) is

$$y - k = \frac{3h}{k}(x - h)$$

$\because$ It passes through the point $\left(\frac{4}{3}, 0\right)$

$$\therefore \quad 0 - k = \frac{3h}{k}\left(\frac{4}{3} - h\right)$$

$$\Rightarrow \quad -k^2 = 4h - 3h^2$$

$$\Rightarrow \quad 3h^2 - k^2 - 4h = 0 \quad \ldots(2)$$

$\because (h, k)$ lies on equation (1),

$$\therefore \quad 3h^2 - k^2 = 8 \quad \ldots(3)$$

From (2) and (3), we get

$$4h = 8$$

$$\Rightarrow \quad h = 2$$

$\therefore$ From (3),

$$3(2)^2 - k^2 = 8$$

$$\Rightarrow \quad k^2 = 4$$

$$\Rightarrow \quad k = \pm 2$$

Hence the points of contact are (2, 2) and (2, – 2).

Equation of tangent at (2, 2) is

$$y - (2) = \frac{3(2)}{2}(x - 2)$$

$$\Rightarrow \quad y - 2 = 3x - 6$$

$$\Rightarrow \quad y = 3x - 4$$

Equation of tangent at (2, – 2) is

$$y - (-2) = \frac{3(2)}{-2}(x - 2)$$

$$\Rightarrow \quad y + 2 = -3(x - 2)$$

$$\Rightarrow \quad y + 2 = -3x + 6$$

$$\Rightarrow \quad y = -3x + 4$$

Q. 18. Find the equation of normal at a point on the curve $x^2 = 4y$, which passes through the point (1, 2). Also, find the equation of the corresponding tangent. *(CBSE, 2013)*

Solution :

The equation of the curve is

$$x^2 = 4y$$

Differentiating w.r.t. x, we get

$$2x = 4\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{2}$$

Let the point of contact be (h, k), then

$$\left.\frac{dy}{dt}\right|_{at\,(h,k)} = \frac{h}{2}$$

$\therefore$ Slope of the tangent at $(h, k) = \frac{h}{2}$

$\therefore$ Slope of the tangent at $(h, k) = -\frac{1}{h/2}$

$$= -\frac{2}{h}$$

$\therefore$ Equation of normal at (h, k) is

$$y - k = -\frac{2}{h}(x - h)$$

$\because$ It passes through the point (1, 2)

$$\therefore \quad 2 - k = -\frac{2}{h}(1 - h)$$

$$\Rightarrow \quad k = 2 + \frac{2}{h}(1 - h) \quad \ldots(1)$$

$\because (h, k)$ lies on the curve $x^2 = 4y$

$$\therefore \quad h^2 = 4k \quad \ldots(2)$$

Solving equations (1) and (2), we get

$$h^2 = 4\left[2 + \frac{2}{h}(1 - h)\right]$$

$$\Rightarrow \quad h^2 = 4\left(\frac{2h + 2 - 2h}{h}\right)$$

$$\Rightarrow \quad h^3 = 8$$

$$\Rightarrow \quad h = 2$$

$\therefore$ from (1), $\quad k = 2 + \frac{2}{2}(1 - 2) = 1$

$\therefore$ The point of contact is (2, 1).

$\therefore$ Equation of normal is

$$y - 1 = -\frac{2}{2}(x - 2)$$

$$\Rightarrow \quad y - 1 = -x + 2$$

$$\Rightarrow \quad x + y = 3$$

Also, equation of the corresponding tangent is

$$y - k = \frac{h}{2}(x - h)$$

$$\Rightarrow \quad y - 1 = \frac{2}{2}(x - 2)$$

$$\Rightarrow \quad y - 1 = x - 2$$

$$\Rightarrow \quad y = x - 1$$

Q. 19. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Solution :

The given curves are

$$x = y^2 \quad \ldots(1)$$

and $\quad xy = k \quad \ldots(2)$

Solving (1) and (2), we get

$$y^3 = k$$

$$\Rightarrow \quad y = k^{1/3}$$

$$\therefore \quad x = k^{2/3}$$

$\therefore$ The point of intersection is $(k^{2/3}, k^{1/3})$.

Differentiating (1) w.r.t. x., we get

$$1 = 2y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$$

$\therefore$ Slope of tangent at $(k^{2/3}, k^{1/3}) = \frac{1}{2k^{1/3}} = m_1$ (say)

Differentiating (2) w.r.t. x., we get

$$x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

$\because$ Slope of tangent at $(k^{2/3}, k^{1/3})$

$$= -\frac{k^{1/3}}{k^{2/3}} = -\frac{1}{k^{1/3}} = m_2 \text{ (say)}$$

The curves (1) and (2) will cut at right angles if

$$m_1 m_2 = -1$$

$$\Rightarrow \left(\frac{1}{2k^{1/3}}\right)\left(-\frac{1}{k^{1/3}}\right) = -1$$

$$\Rightarrow \frac{1}{2k^{\frac{1}{3}}} = 1$$

$$\Rightarrow 2k^{2/3} = 1$$

$$\Rightarrow 8k^2 = 1 \quad \text{(cubing both sides)}$$

Q. 20. Prove that all normals to the curve $x = a \cos t + at \sin t$, $y = a \sin t - at \cos t$ are at a constant distance 'a' from the origin. [*CBSE*, 2013 (*Comptt.*)]

Solution :

$$x = a \cos t + at \sin t$$

$$\Rightarrow \frac{dx}{dt} = -a \sin t + a \sin t + at \cos t = at \cos t$$

and

$$y = a \sin t - at \cos t$$

$$\Rightarrow \frac{dy}{dt} = a \cos t - a \cos t + at \sin t = at \sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{at \sin t}{at \cos t} = \tan t$$

$\therefore$ Slope of the tangent $= \tan t$

$\therefore$ Slope of the normal $= -\dfrac{1}{\tan t} = -\cot t$

$\therefore$ Equation of the normal is

$$y - (a \sin t - at \cos t) = -\cot t \{x - (a \cos t + at \sin t)\}$$

$$\Rightarrow y \sin t - a \sin^2 t + at \sin t \cot t = -x \cos t + a \cos^2 t + at \sin t \cos t$$

$$\Rightarrow x \cos t + y \sin t = a \quad \ldots(1)$$

Length of the perpendicular from equation (1)

$$= \left|\frac{0 \cos t + 0 \sin t - a}{\sqrt{\cos^2 t + \sin^2 t}}\right|$$

$= a =$ constant

Long Answer Type Questions

Q. 1. AB is a diameter of a circle and C is any point on the circle. Show that the area of Δ ABC is maximum, when it is isosceles.

[*AI CBSE*, 2014 (*Comptt.*)]

Solution :

Let the radius of the circle be r, then AB = $2r$

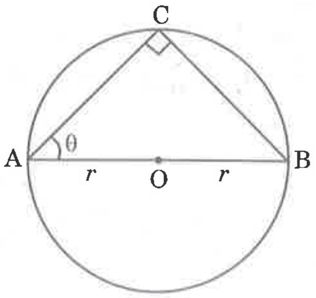
$\because$ C lies on the circle

$\therefore \angle ACB = 90°$

Let $\angle CAB = \theta$

then $AC = 2r \cos \theta$

and $BC = 2r \sin \theta$



Let S be the area of Δ ABC, then

$$S = \frac{1}{2}.AC.BC$$

$$= \frac{1}{2}(2r \cos \theta)(2r \sin \theta)$$

$$= 2r^2 \sin \theta \cos \theta$$

$$= r^2 \sin 2\theta$$

S is maximum when $r^2 \sin 2\theta$ is maximum.

$\Rightarrow \sin 2\theta$ is maximum

$$\Rightarrow \sin 2\theta = 1 = \sin 90°$$

$$\Rightarrow 2\theta = 90°$$

$$\Rightarrow \theta = 45°$$

$$\therefore AC = 2r \cos 45° = 2r.\frac{1}{\sqrt{2}} = \sqrt{2}\,r$$

and

$$BC = 2r \sin 45° = 2r.\frac{1}{\sqrt{2}} = \sqrt{2}\,r$$

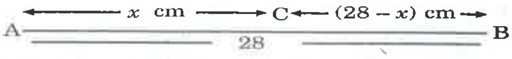
$\therefore$ AC = BC

$\therefore$ Δ ABC is isosceles.

$\therefore$ Area of Δ ABC is maximum, when it is isosceles.

Q. 2. A wire of length 28 cm is to be cut into two pieces. One of the piece is to be made into a square and other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum ? (*BSER*, 2013)

Solution :



Let AB be a wire of length 28 cm. Let the wire AB be cut into two pieces at point C.

Let AC = x cm

then BC = $(28 - x)$ cm

Let the piece AC be made into a square of side a.

then $4a = x$

$$\Rightarrow a = \frac{x}{4}$$

$\therefore$ Area of the square $= a^2 = \left(\dfrac{x}{4}\right)^2 = \dfrac{x^2}{16}$ cm^2

Let the piece BC be made into a circle of radius r, then

$$2\pi r = 28 - x$$

$$\Rightarrow r = \frac{28 - x}{2\pi}$$

$\therefore$ Area of the circle $= \pi r^2 = \pi\left(\dfrac{28 - x}{2\pi}\right)^2$ cm^2

$$= \frac{(28 - x)^2}{4\pi} \text{ cm}^2$$

Let A be the combined area of the square and the circle, then

$$A = \frac{x^2}{16} + \frac{(28 - x)^2}{4\pi}$$

For A to be maximum or minimum,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \frac{2x}{16} + \frac{2(28 - x)}{4\pi}(-1) = 0$$

$$\Rightarrow \frac{x}{8} - \frac{28 - x}{2\pi} = 0$$

$$\Rightarrow \frac{x}{8} - \frac{14}{\pi} + \frac{x}{2\pi} = 0$$

$$\Rightarrow x\left(\frac{1}{8} + \frac{1}{2\pi}\right) = \frac{14}{\pi}$$

$$\Rightarrow x\frac{(\pi + 4)}{8\pi} = \frac{14}{\pi}$$

$$\Rightarrow x = \frac{112}{\pi + 4}$$

Also, $\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{1}{2\pi} (> 0)$

$\therefore$ A is minimum when $x = \frac{112}{\pi + 4}$

Hence the wire should be cut at C such that

AC = $\frac{112}{\pi + 4}$ cm.

Q. 3. Find the equation of the normal to curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

[*CBSE, Delhi*, 2010, 13 (*Comptt.*)]

Solution :

Equation of given curve is : $y = x^3 + 2x + 6$ ...(i)

Differentiating w.r.t. x, we get

$$\left.\begin{aligned}\frac{dy}{dx} &= 3x^2 + 2\\ \therefore \text{Slope of tangent} &= 3x^2 + 2\end{aligned}\right\}$$

$\Rightarrow$ Slope of normal $= -\frac{1}{3x^2 + 2}$

Equation of given line $x + 14y + 4 = 0$ ...(ii)

Differentiate w.r.t. x, we get

$$1 + 14\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{14}$$

$\therefore$ Slope of line (ii) $= -\frac{1}{14}$

Now given that the normal is parallel to the line (ii).

$\therefore$ Slope of normal = Slope of line (ii)

i.e., $-\frac{1}{3x^2 + 2} = -\frac{1}{14}$ and $3x^2 + 2 = 14$

$\Rightarrow$ $3x^2 = 12$

$\Rightarrow$ $x = \pm 2$

when $x = 2$, $y = (2)^3 + 2(2) + 6 = 18$

and when $x = -2$, $y = (-2)^3 + 2(-2) + 6 = -6$

Equation of normal at P(2, 18) :

$$y - 18 = -\frac{1}{14}(x - 2)$$

$$\left[\because \text{Slope of normal} = -\frac{1}{14}\right]$$

$\Rightarrow$ $14y - 252 = -x + 2$

$\Rightarrow$ $x + 14y - 254 = 0$

Equation of normal at Q (– 2, – 6) :

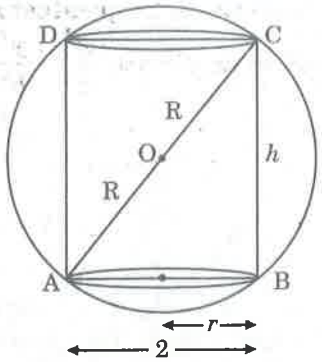
$$y + 6 = -\frac{1}{14}(x + 2)$$

$\Rightarrow$ $14y + 84 + x + 2 = 0$

$\Rightarrow$ $x + 14y + 86 = 0$

Q. 4. Show that the height of the cylinder of greatest volume which can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

(*USEB, AI CBSE*, 2014)



Solution :

Let r and h be the radius and height respectively of the inscribed cylinder.

$\therefore$ In right triangle ABC,

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow$ $(2R)^2 = (2r)^2 + h^2$

$\Rightarrow$ $4R^2 = 4r^2 + h^2$ ...(1)

Let V be the volume of the cylinder, then

$$V = \pi r^2 h$$

$\Rightarrow$ $V = \pi\left(\frac{4R^2 - h^2}{4}\right)h$ [Using (1)]

$\Rightarrow$ $V = \pi R^2 h - \frac{\pi h^3}{4}$

For maximum or minimum values of V,

$$\frac{dV}{dh} = 0$$

$\Rightarrow$ $\pi R^2 - \frac{3\pi h^2}{4} = 0$

$\Rightarrow$ $R^2 = \frac{3h^2}{4}$

$\Rightarrow$ $h^2 = \frac{4}{3}R^2$

$\Rightarrow$ $h = \frac{2}{\sqrt{3}}R$

Also, $\frac{d^2V}{dh^2} = -\frac{6\pi h}{4} = -\frac{3\pi h}{2}$

$\therefore$ $\left.\frac{d^2v}{dh^2}\right|_{h=\frac{2}{\sqrt{3}}R} = -\frac{3\pi}{2}\cdot\frac{2}{\sqrt{3}}R$

$= -\sqrt{3}\,\pi R\ (< 0)$

$\therefore$ V is maximum when $h = \frac{2}{\sqrt{3}}R$

Maximum volume $= \pi r^2 h$

$$= \pi\left(\frac{4R^2 - h^2}{4}\right)h$$

$$= \frac{\pi}{4}\left(4R^2 - \frac{4}{3}R^2\right)\frac{2}{\sqrt{3}}R$$

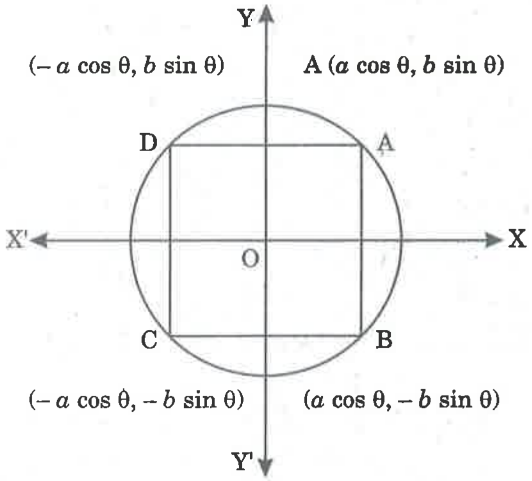
$$= \frac{\pi}{4}\cdot\frac{8R^2}{3}\cdot\frac{2}{\sqrt{3}}R$$

$$= \frac{4\pi R^3}{3\sqrt{3}}$$

Q. 5. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(*AI CBSE*, 2013)

Solution :



Let A (a cos θ, b sin θ) be a corner of the inscribed rectangle ABCD, then

B → (a cos θ, – b sin θ)
C → (– a cos θ, – b sin θ)
D → (– a cos θ, b sin θ)

∴ AD = $2a\cos\theta$
and AB = $2b\sin\theta$

∴ Area of rectangle ABCD

$$= AD \times AB$$
$$= (2a\cos\theta)(2b\sin\theta)$$
$$= 2ab\sin 2\theta$$

∴ Area of rectangle ABCD is the greatest when $2ab\sin 2\theta$ is greatest

⇒ $\sin 2\theta$ is greatest

⇒ $\sin 2\theta = 1 = \sin 90°$

⇒ $2\theta = 90°$

⇒ $\theta = 45°$

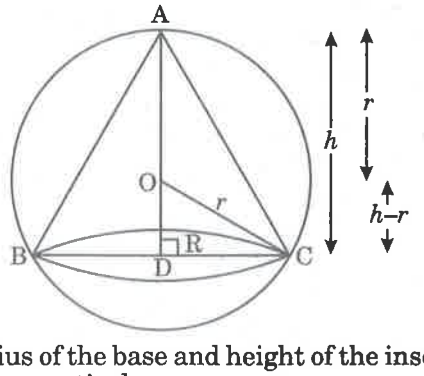
∴ Area of the greatest rectangle

$$= 2ab$$

Q. 6. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

(*CBSE*, 2013; *AI CBSE*, 2014; (*Comptt.*)]

Solution :



Let radius of the base and height of the inscribed cone be R and h respectively.

then OD = $h - r$

In right triangle ODC,

$$r^2 = R^2 + (h-r)^2$$

⇒ $r^2 = R^2 + h^2 + r^2 - 2hr$

⇒ $R^2 = 2hr - h^2$ ...(1)

Let V be the volume of the cone, then

$$V = \frac{1}{3}\pi R^2 h$$
$$= \frac{1}{3}\pi(2hr - h^2)h \quad \text{[From (1)]}$$
$$= \frac{1}{3}\pi(2h^2r - h^3)$$

⇒ $\frac{dV}{dh} = \frac{1}{3}\pi(4hr - 3h^2)$

For maxima or minima of V

$$\frac{dV}{dh} = 0$$

⇒ $\frac{1}{3}\pi(4hr - 3h^2) = 0$

⇒ $4hr - 3h^2 = 0$

⇒ $4hr - 3h = 0$ $(\because h \neq 0)$

⇒ $h =$

Also, $\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 6h)$

at $h = \frac{4V}{3}$, $\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 8r)$

$$= -\frac{4\pi r}{3} < 0.$$

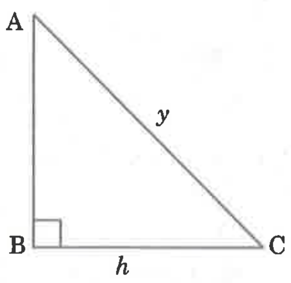
∴ V is maximum at $h = \frac{4r}{3}$.

Maximum value $= \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi(2h^2r - h^3)$

$$= \frac{1}{3}\pi\left\{2\frac{16r^2}{9}.r - \frac{64r^3}{27}\right\}$$
$$= \frac{1}{3}\pi\left\{\frac{32r^3}{9} - \frac{64r^3}{27}\right\} = \frac{32\pi r^3}{81}$$
$$= \frac{8}{27}\left(\frac{4}{3}\pi r^3\right) = \frac{8}{27}$$

(Volume of sphere)

Q. 7. If the sum of the lengths of the hypotenuse and a side of a right triangle is given, show that the area of the triangle is maximum when the angle between them is 60°. (*AI CBSE*, 2014)



Solution : Let ABC be a right triangle right angled at B.

Let hypotenuse AC = y and a side BC = x,

then $y + x = k$ (given)...(1)

Let A be the area of ΔABC, then

$$A = \frac{1}{2} BC \times AB$$
$$= \frac{1}{2} x \sqrt{y^2 - x^2}$$
$$= \frac{1}{2} x \sqrt{(k-x)^2 - x^2} \quad \text{[From (1)]}$$
$$= \frac{1}{2} x \sqrt{k^2 - 2kx}$$

$\Rightarrow$ $$A^2 = \frac{x^4}{4}(k^2 - 2kx)$$

$\Rightarrow$ $$4A^2 = k^2x^2 - 2kx^3$$

$\Rightarrow$ $$Z = k^2x^2 - 2kx^3$$

If A is maximum or minimum, then Z is maximum or minimum.

For maxima or minima of Z,

$$\frac{dZ}{dx} = 0$$

$\Rightarrow$ $2k^2x - 6kx^2 = 0$

$\Rightarrow$ $2k(kx - 3x^2) = 0$

$\Rightarrow$ $2kx(k - 3x) = 0$

$\Rightarrow$ $x(k - 3x) = 0$

$\Rightarrow$ $x = 0, \frac{k}{3}$

$x = 0$ is inadmissible.

$\therefore$ $$x = \frac{k}{3}$$

$$\frac{d^2Z}{dx^2} = 2k^2 - 2kx$$

At $x = \frac{k}{3}$,

$$\frac{d^2Z}{dx^2} = 2k^2 - 12k\frac{k}{3}$$
$$= -2k^2 \ (< 0)$$

$\therefore$ Z is maximum at $x = \frac{k}{3}$

$\Rightarrow 4A^2$ is maximum at $x = \frac{k}{3}$

$\Rightarrow A^2$ is maximum at $x = \frac{k}{3}$

$\Rightarrow$ A is maximum at $x = \frac{k}{3}$

$\therefore$ $$y = k - x = k - \frac{k}{3} \quad \text{[From (1)]}$$
$$= \frac{2k}{3}$$

Now, from right triangle ABC,

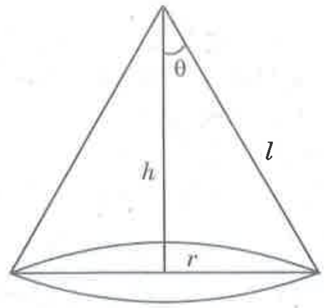
$$\cos C = \frac{x}{y} = \frac{k/3}{2k/3}$$
$$= \frac{1}{2} = \cos 60^\circ$$

$\Rightarrow$ $C = 60^\circ$

Q. 8. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface is $\cot^{-1}\sqrt{2}$. *(CBSE, 2014)*

Solution :

Let r be the base radius, h be the vertical height, l be the slant height and θ be the semi-vertical angle of the cone. Let V be the volume and A be the curved surface area of the cone, then



$$V = \frac{1}{3}\pi r^2 h \quad \ldots(1)$$
$$A = \pi r l$$

$\Rightarrow$ $$A = \pi r\sqrt{r^2 + h^2} \quad \ldots(2)$$

From (1), $h = \frac{3V}{\pi r^2}$ where V is given. ...(3)

Using (3), (2) gives,

$$A = \pi r \sqrt{r^2 + \left(\frac{3V}{\pi r}\right)^2}$$

$\Rightarrow$ $$A = \pi r \sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}}$$

$\Rightarrow$ $$A^2 = \pi^2 r^2 \sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}}$$

$\Rightarrow$ $$A^2 = \pi^2 r^4 + \frac{9V^2}{r^2}$$

$\Rightarrow$ $$Z = \pi^2 r^4 + \frac{9V^2}{r^2}, \text{where } Z = A^2$$

$\therefore$ A is least, then A^2 is least, *i.e.*, Z is least.

For maxima or minima of Z,

$$\frac{dZ}{dr} = 0$$

$\Rightarrow$ $$4\pi^2 r^3 - \frac{18V^2}{r^3} = 0$$

$\Rightarrow$ $$4\pi^2 r^3 - \frac{18}{r^3}\left(\frac{1}{3}\pi r^2 h\right)^2 = 0$$

$\Rightarrow$ $$4\pi^2 r^3 - \frac{18}{r^3}\frac{1}{9}\pi^2 r^4 h^2 = 0$$

$\Rightarrow$ $4\pi^2 r^3 - 2\pi^2 h^2 r = 0$

$\Rightarrow$ $2\pi^2 r(2r^2 - h^2) = 0$

$\Rightarrow$ $r(2r^2 - h^2) = 0$

$\Rightarrow$ $r = 0, r^2 = \frac{h^2}{2}$

$r = 0$ is inadmissible,

$\therefore$ $$r^2 = \frac{h^2}{2}$$

also, $$\frac{d^2Z}{dr^2} = 12\pi^2 r^2 + \frac{54V^2}{r^4}$$

at $r^2 = \frac{h^2}{2}$, $$\frac{d^2Z}{dr^2} = 12\pi^2\frac{h^2}{2} + \frac{54V^2}{h^4} \quad (4)$$
$$= 6\pi^2 h^2 + \frac{216V^2}{h^4} \quad (> 0)$$

$\therefore$ Z is least when $r^2 = \frac{h^2}{2}$

$\Rightarrow A^2$ is least when $r^2 = \frac{h^2}{2}$

$\Rightarrow$ A is least when $r^2 = \frac{h^2}{2}$

Now, $r^2 = \frac{h^2}{2}$

$\Rightarrow \quad \frac{h^2}{r^2} = 2$

$\Rightarrow \quad \frac{h}{r} = \sqrt{2}$

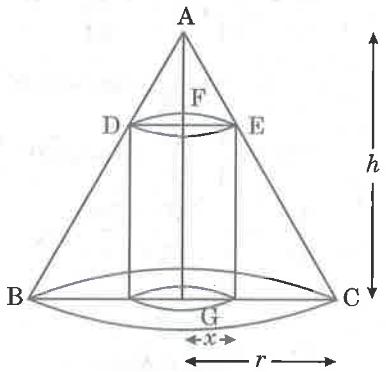
$\Rightarrow \quad \cot\theta = \sqrt{2}$

$\Rightarrow \quad \theta = \cot^{-1}(\sqrt{2})$

Q. 9. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half that of the cone.

[*CBSE*, 2013 (*Comptt.*)]

Solution :



Let the base radius and height of the cone be r and h respectively. Let the base radius and height of the inscribed cylinder be x and H respectively.

$\because \quad \Delta AFE \sim \Delta AGC$

$\therefore \quad \frac{AF}{AG} = \frac{FE}{GC}$

$\Rightarrow \quad \frac{h - H}{h} = \frac{x}{r}$

$\Rightarrow \quad h - H = \frac{hx}{r}$

$\Rightarrow \quad H = h - \frac{hx}{r}$

$\Rightarrow \quad H = \frac{h(r-x)}{r}$

Let S be the curved surface area of the inscribed cylinder, then

$S = 2\pi x H$

$\Rightarrow \quad S = 2\pi x \frac{h(r-x)}{r}$

$\Rightarrow \quad S = \frac{2\pi h}{r}(rx - x^2)$

For maxima or minima of S,

$$\frac{dS}{dx} = 0$$

$\Rightarrow \quad \frac{2\pi h}{r}(r - 2x) = 0$

$\Rightarrow \quad r - 2x = 0$

$\Rightarrow \quad x = \frac{r}{2}$

then, $\frac{d^2S}{dx^2} = -\frac{4\pi h}{r} \ (< 0)$

$\therefore$ S is maximum when $x = \frac{r}{2}$.

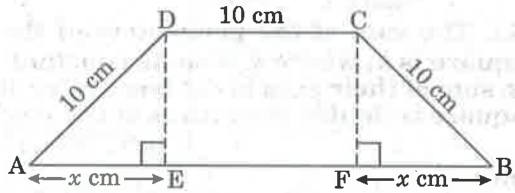
$\therefore$ Radius of the inscribed cylinder

$= \frac{1}{2}$ Radius of the cone.

Q. 10. If the length of three sides of a trapezium other than base is 10 cm each, then find the area of the trapezium when it is maximum.

[*AI CBSE*, 2014 (*Comptt.*)]

Solution :



Let ABCD be a trapezium in which

AD = DC = CB = 10 cm

Draw DE $\perp$ AB and CF $\perp$ AB

Let AE = x cm

then FB = x cm

In right triangle CFB,

$CF^2 + FB^2 = CB^2$ (By pythagoras theorem)

$\Rightarrow \quad CF^2 + x^2 = 10^2$

$\Rightarrow \quad CF = \sqrt{100 - x^2}$

$\Rightarrow \quad DE = CF = \sqrt{100 - x^2}$ cm

$\therefore$ Area of the trapezium (S)

$= \frac{1}{2}\{10 + (10 + 2x)\}\sqrt{100 - x^2}$

$\Rightarrow \quad S = (10 + x)\sqrt{100 - x^2}$

$\Rightarrow \quad S^2 = (10 + x)^2(100 - x^2)$ (squaring)

$\Rightarrow \quad Z = (10+x)^2(100-x^2)$, where $Z = S^2$

S is maximum.

$\Rightarrow S^2$ is maximum.

$\Rightarrow$ Z is maximum.

For maxima or minima of Z,

$$\frac{dZ}{dx} = 0$$

$\Rightarrow \quad (10 + x)^2(-2x) + 2(10 + x)(100 - x^2) = 0$

$\Rightarrow \quad (10 + x)^2(-2x) + 2(10 + x)(10 + x)(10 - x) = 0$

$\Rightarrow \quad 2(10 + x)^2\{-x + 10 - x\} = 0$

$\Rightarrow \quad 2(10 + x)^2(10 - 2x) = 0$

$\Rightarrow \quad (10 + x)^2(10 - 2x) = 0$

$\Rightarrow \quad 10 + x = 0$ or $10 - 2x = 0$

$\Rightarrow \quad x = -10$ or $x = 5$

$\Rightarrow \quad x = -10, 5$

$\because x = -10$ is inadmissible,

$\therefore \quad x = 5$

Also, $\frac{d^2Z}{dx^2} = 2.2.(10+x)(10-2x) + 2(10+x)^2(-2)$
$= 4(10+x)(10-2x-10-x)$
$= 4(10+x)(-3x)$
$= -12x(10+x)$

At $x = 5$, $\frac{d^2Z}{dx^2} = -12(5)(10+5)$
$= -900 (< 0)$

$\therefore$ Z is maximum when $x = 5$

$\therefore$ $Z_{maximum} = (10+5)^2(100-5^2)$
$= (15)^2(75)\ cm^2$

$\Rightarrow$ $S^2_{maximum} = (15)^2(75)\ cm^2$

$\Rightarrow$ $S_{maximum} = 15\sqrt{75}\ cm^2$

$\Rightarrow$ $S_{maximum} = 75\sqrt{3}\ cm^2$

Q. 11. The sum of the perimeters of the circle and a square is k, where k is some constant. Prove that the sum of their area is the least when the side of the square is double the radius of the circle.
[*CBSE*, 2014 (*Comptt.*)]

Solution :

Let r be the radius of the circle and x be the side of the square, then

$$4x + 2\pi r = k \qquad ...(1)$$

Let S be the sum of the area of the circle and the square, then

$$S = \pi r^2 + x^2$$

$\Rightarrow$ $S = \pi r^2 + \left(\frac{k-2\pi r}{4}\right)^2$ [Using (1)]

For maxima or minima of S,

$$\frac{dS}{dr} = 0$$

$\Rightarrow$ $2\pi r + \frac{2}{16}(k-2\pi r)(-2\pi) = 0$

$\Rightarrow$ $2\pi r - \frac{\pi(k-2\pi r)}{4} = 0$

$\Rightarrow$ $2\pi r - \frac{\pi k}{4} + \frac{\pi^2 r}{2} = 0$

$\Rightarrow$ $r\left(2\pi + \frac{\pi^2}{2}\right) = \frac{\pi k}{4}$

$\Rightarrow$ $r\pi\left(2 + \frac{\pi}{2}\right) = \frac{\pi k}{4}$

$\Rightarrow$ $r\left(2 + \frac{\pi}{2}\right) = \frac{k}{4}$

$\Rightarrow$ $r\left(\frac{4+\pi}{2}\right) = \frac{k}{4}$

$\Rightarrow$ $r = \frac{k}{2(4+\pi)}$

Also, $\frac{d^2S}{dr^2} = 2\pi - \frac{4\pi}{16}(-2\pi)$
$= 2\pi + \frac{\pi^2}{2} > 0$

$\Rightarrow$ S is minimum when $r = \frac{k}{2(4+\pi)}$

$\therefore$ From (1),

$x = \frac{k-2\pi r}{4}$
$= \frac{2r(4+\pi)-2\pi r}{4}$
$= 2r$

$\therefore$ S is least when the side of the square is double the radius of the circle.

Q. 12. A ballon which always remains spherical is being inflated by pumping in 900 cubic centimetre of gas per second. Find the rate at which the radius of the ballon is increasing when the radius is 15 cm.
(*JAC*, 2015)

Solution : Let the radius of sphere is r, then

$$\text{Volume V} = \frac{4}{3}.\pi r^3$$

differentiate *w.r.*to t

$$\frac{dv}{dt} = \frac{4}{3}\pi \times 3r^2\frac{dr}{dt}$$

$\left(\because \frac{dv}{dt} = 900\ cm^3 \text{ gas per sec and } r = 15\ cm\right)$

$\Rightarrow$ $900 = \frac{4}{3} \times \frac{22}{7} \times 3 \times (15)^2.\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{900 \times 7 \times 3}{4 \times 22 \times 3 \times (15)^2}$
$= \frac{6300}{19800}$
$= 0.32$ cm/sec

Q. 13. The radius of a circle is changing uniformly at the rate of 3 cm/sec. Find the rate of change of its area when its radius is 10 cm.
(*USEB*, 2015)

Solution : Let the radius of circle is r and $r = 10$ cm then change in radius

$$\frac{dr}{dt} = 3 \text{ cm/sec.}$$

Area of circle

$$A = \pi r^2$$

difference *w.r.*to t,

$\frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$
$= 2 \times \frac{22}{7} \times 10 \times 3$
$= 188.57\ cm^2$/sec.

NCERT QUESTIONS

Q. 1. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangent are ∥ to x-axis. (*CBSE*, 2011)

Solution :

$$x^2 + y^2 - 2x - 3 = 0$$

$\Rightarrow$ $2x + 2y\frac{dy}{dx} - 2 = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-x}{y}$$

Tangent line is parallel to x-axis.

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1-x}{y} = 0 \Rightarrow x = 1$$

At $x = 1$,

$$1 + y^2 - 2 - 3 = 0$$

$$y^2 = 4, y = \pm 2$$

Hence the points are $(1, \pm 2)$.

Q. 2. Sand pouring from a pipe at the rate of 12 cm³/sec. The falling sand form a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of base. How fast is the height of the sand cone increasing when the height is 4 cm ? *(CBSE, 2011)*

Solution :

$$\frac{dv}{dt} = 12 \text{ cm}^3/\text{sec} \quad \ldots(1)$$

and $$h = \frac{1}{6} r \Rightarrow r = 6h$$

Now $$v = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad v = \frac{1}{3}\pi(36h^2)h$$

$$\therefore \quad v = 12\pi h^3$$

$$\Rightarrow \quad \frac{dv}{dt} = 36\pi h^2 . \frac{dh}{dt}$$

$$\Rightarrow \quad 12 = 36\pi \times (4)^2 \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36 \times 16\pi}$$

$$= \frac{1}{48\pi} \text{ cm/sec.}$$

Q. 3. Find the eqn. of all lines having slope 0 which are tangent to the curve $y = \dfrac{1}{x^2 - 2x + 3}$.

Solution :

$$y = \frac{1}{x^2 - 2x + 3}$$

$$\frac{dy}{dx} = -\frac{(2x-2)}{(x^2 - 2x + 3)^2}$$

$\because$ Slope of line is 0.

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad (2x - 2) = 0$$

$$\Rightarrow \quad x = 1$$

At $x = 1$, $$y = \frac{1}{1-2+3} = \frac{1}{2}$$

Hence the point of contact is $\left(1, \dfrac{1}{2}\right)$.

Equation of tangent line at the point (1, 1/2) is

$$\frac{y - 1/2}{x - 1} = \frac{0}{1}$$

$$\Rightarrow \quad y - 1/2 = 0$$

$\Rightarrow$ Equation of line is $y = 1/2$.

Q. 4. Find the equation of the tangent and normal to the hyperbola, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Solution :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots(1)$$

$$\Rightarrow \quad \frac{2x}{a^2} - \frac{2y^2}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

$$\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = \frac{b^2 x_0}{a^2 y_0}$$

Equation of tangent line is $\dfrac{y - y_0}{x - x_0} = \dfrac{b^2 x_0}{a^2 y_0}$

$$\Rightarrow \quad a^2 y y_0 - a^2 y_0^2 = b^2 x x_0 - b^2 x_0^2$$

$$\Rightarrow \quad b^2 x x_0 - a^2 y y_0 = b^2 x_0^2 - a^2 y_0^2$$

$$\Rightarrow \quad \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}$$

From the equation of curve, $\dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$

$$\Rightarrow \quad \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$$

Equation of normal at (x_0, y_0) is

$$y - y_0 = -\frac{1}{\left(\dfrac{b^2 x_0}{a^2 y_0}\right)}(x - x_0)$$

$$\Rightarrow \quad y - y_0 = -\frac{a^2 y_0}{b^2 x_0}(x - x_0)$$

$$\Rightarrow \quad \frac{y - y_0}{a^2 y_0} + \frac{x - x_0}{b^2 x_0} = 0.$$

Q. 5. Of all the closed cylindrical cones of given volume of 100 cm³. Find dimensions of cone if surface area is minimum.

Solution :

According to question,

$$\pi r^2 h = 100 \quad \ldots(1)$$

$$S = 2\pi r h + 2\pi r$$

$$\Rightarrow \quad S = 2\pi r\left(\frac{100}{\pi r^2}\right) + 2\pi r^2$$

$$\Rightarrow \quad S = \frac{200}{r} + 2\pi r^3$$